REPRESENTATION THEORY FOR DILUTE LATTICE MODELS

BRUCE W. WESTBURY

Contents

1. Introduction

In this paper we study the representation theory associated to dilute lattice models. Given a solvable lattice model then often there are dilute versions of the model which are also solvable. These dilute models are studied in [Roc92], [GP93], [Gri94] and [WN93]. In this paper we discuss the algebras generated by the single bond transfer matrices and their representation theory.

Our motivation for studying these dilute models is that in [DPT03] it is shown that there is a connection between the dilute Potts models and the exceptional series of Lie algebras proposed in [Del96], [DdM96] and [CdM96].

2. Categories

In this section we give the main construction of this paper. This construction gives a tensor product of monoidal categories. Before giving this construction we make some remarks on monoidal categories.

In this paper nearly all the monoidal categories are K-linear for some integral domain K and furthermore these two structures are compatible. In practice, it is always the case that $K = \mathrm{End}(1)$.

If $\mathcal{C}$ is a category then we can construct $K\mathcal{C}$, the free K-linear category on $\mathcal{C}$. The universal property of this construction is that it gives a left adjoint to the forgetful functor from K-linear categories to categories. This construction passes to monoidal categories.

Date: February 1, 2008.

Another useful property of a $\mathbb{Z}$-linear category is the additive property. This property says that we can form the direct sum of two (or more) objects and that there is a zero object. If $\mathcal{C}$ is a $\mathbb{Z}$-linear category then we can construct a free additive category on $\mathcal{C}$ by taking objects to be vectors of objects in $\mathcal{C}$ and morphisms to be matrices of morphisms. The universal property of this construction is that it gives a left adjoint to the forgetful functor from additive categories to $\mathbb{Z}$-linear categories. This construction passes to monoidal categories.

The following construction defines a tensor product of categories. This construction gives a category which is equivalent to the construction in [BK01, 1.1.15]. Both of these constructions are tensor products in the sense of [Del90, §5].

Definition 2.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two monoidal categories. Then we construct a new monoidal category, $\mathcal{C}_1 \otimes \mathcal{C}_2$ as follows. The objects are sequences of elements of the disjoint union $\mathrm{Ob}(\mathcal{C}_1) \cup \mathrm{Ob}(\mathcal{C}_2)$. Let X be any such sequence. Then associated to X is a sequence X_1 of elements of $\mathrm{Ob}(\mathcal{C}_1)$ and a sequence X_2 of elements of $\mathrm{Ob}(\mathcal{C}_2)$. Then given two sequences X and Y we define a morphism $\phi\colon X \to Y$ to be a pair (ϕ_1, ϕ_2) where $\phi_1\colon \otimes_{x\in X_1} x \to \otimes_{y\in Y_1} y$ is a morphism in $\mathcal{C}_1$ and similarly $\phi_2\colon \otimes_{x\in X_2} x \to \otimes_{y\in Y_2} y$ is a morphism in $\mathcal{C}_2$. The tensor product is defined on objects by concatenating sequences and is defined on morphisms by taking the tensor products in $\mathcal{C}_1$ and $\mathcal{C}_2$.*

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both K-linear monoidal categories then we modify this construction so that it gives another K-linear category. In this version we take

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}_{\mathcal{C}_1}(\otimes_{x\in X_1} x, \otimes_{y\in Y_1} y) \otimes \mathrm{Hom}_{\mathcal{C}_2}(\otimes_{x\in X_2} x, \otimes_{y\in Y_2} y)$$

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are both K-linear and additive monoidal categories then we take the free additive category on this to give a monoidal category which is also K-linear and additive.

The first property of this construction is that it is associative. This means that we natural equivalences of monoidal categories

$$(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3 \sim \mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3) \tag{1}$$

The simplest way to see this is to note that both sides are naturally equivalent to a monoidal category whose objects are sequences of elements of $\mathrm{Ob}(\mathcal{C}_1) \cup \mathrm{Ob}(\mathcal{C}_2) \cup \mathrm{Ob}(\mathcal{C}_3)$.

A further property is that it is symmetric. This structure is given by natural equivalences of categories

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \cong \mathcal{C}_2 \otimes \mathcal{C}_1 \tag{2}$$

This functor is given by

$$(\phi_1, \phi_2) \mapsto (\phi_2, \phi_1)$$

Note that if H_1 and H_2 are Hopf algebra and $\mathrm{Rep}(H_1)$ and $\mathrm{Rep}(H_2)$ are monoidal categories of representations then we have a natural functor

$$\mathrm{Rep}(H_1) \otimes \mathrm{Rep}(H_2) \to \mathrm{Rep}(H_1 \otimes H_2)$$

This functor is defined on objects as follows. Each object V of $\mathrm{Rep}(H_1)$ gives an object $V \otimes 1$ of $\mathrm{Rep}(H_1 \otimes H_2)$ and each object V of $\mathrm{Rep}(H_2)$ gives an object $1 \otimes V$ of $\mathrm{Rep}(H_1 \otimes H_2)$. Then each object of $\mathrm{Rep}(H_1) \otimes \mathrm{Rep}(H_2)$ can be regarded as a sequence of objects of $\mathrm{Rep}(H_1 \otimes H_2)$ and we take the tensor product to give an object of $\mathrm{Rep}(H_1 \otimes H_2)$.

Here we recall some standard constructions in a monoidal category.

Definition 2.2. *Let $\mathcal{C}$ be a K-linear monoidal category and V an object of $\mathcal{C}$. Then $\mathcal{I}(V)$ is the category with objects $\{n|n \geq 0\}$ and morphisms given by*

$$\mathrm{Hom}_{\mathcal{I}(V)}(n, m) = \mathrm{Hom}_{\mathcal{C}}(\otimes^n V, \otimes^m V)$$

This is a K-linear monoidal subcategory of $\mathcal{C}$ and is called the category of invariant tensors. Closely related to this is the sequence of algebras $\{A(n)\}$ given by

$$A(n) = \mathrm{End}_{\mathcal{C}}(\otimes^n V) \tag{3}$$

which are the endomorphism algebras of the objects of the category of invariant tensors. This construction is the motivation for the following definition:

Definition 2.3. *A tower of algebras is a sequence of algebras $\{A(n)\}$ together with homomorphisms $\phi_{n,m} : A(n) \otimes A(m) \to A(n + m)$ which satisfy the associativity condition.*

$$\varphi_{r,s+t}(1 \otimes \varphi_{s,t}) = \varphi_{r+s,t}(\varphi_{r,s} \otimes 1)$$

Another way of stating this associativity condition is to say that the following diagram commutes:

$$\begin{array}{ccc}
A(r) \otimes A(s) \otimes A(t) & \xrightarrow{1\otimes\varphi_{s,t}} & A(r) \otimes A(s+t) \\
{\scriptstyle \varphi_{r,s}\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{r,s+t}} \\
A(r+s) \otimes A(t) & \xrightarrow[\varphi_{r+s,t}]{} & A(r+s+t)
\end{array}$$

Then it is clear that the sequence of algebras in (3) is a tower of algebras; this uses the tensor product in $\mathcal{C}$.

We can also regard a tower of algebras as a K-linear monoidal category denoted by $\mathcal{A}$.

Definition 2.4. *The objects of $\mathcal{A}$ are $\{n|n \geq 0\}$ and morphisms are given by*

$$\mathrm{Hom}_{\mathcal{A}}(n, m) = \begin{cases} A(n) & \textit{if } n = m \\ 0 & \textit{otherwise} \end{cases} \tag{4}$$

Let V be an object of a monoidal category $\mathcal{C}$. Then we can take the tower of algebras associated to V in (3) and then construct the category $\mathcal{A}(V)$ by (2.4). Note that we then have an inclusion of K-linear monoidal categories $\mathcal{A}(V) \to \mathcal{I}(V)$.

For applications to knot theory and topological field theory these categories are required to have extra structure. Next we show that these structures pass to the tensor product.

Examples of tensor categories are tensor categories defined by diagrams. The main examples are the Temperley-Lieb category in [Wes95], the category of braids, and the various categories of tangles (tangles, oriented tangles, framed tangles, and oriented framed tangles). Let $\mathcal{D}$ be one of these tensor categories of diagrams. Then consider the repeated tensor product $\otimes^c \mathcal{D}$. This can be considered as a diagram category where a diagram consists of a diagram with strings labelled by elements of C. Then taking the strings labelled by $c \in C$ gives a diagram in $\mathcal{D}$ and strings of different colours can cross. This is illustrated in [GM03]. Then it is observed in [GP93] that there are functors of tensor categories

$$\mathcal{D} \to \otimes^c(\mathbb{Z}\mathcal{D}) \tag{5}$$

for all c. Let D be a diagram in $\mathcal{D}$. Then the result of applying the functor to D is the sum of all possible colourings of D.

The Temperley-Lieb category depends on a parameter whereas the other examples do not involve a parameter. Next we explain how this parameter behaves when this construction is applied to Temperley-Lieb categories. Let $\mathcal{T}(\delta)$ be the Temperley-Lieb category with parameter δ. Then note that this construction gives functors

$$\mathcal{T}(\sum_{c\in C} \delta_c) \to \otimes_{c\in C} \mathcal{T}(\delta_c)$$

3. Basic Example

Next we take a very simple monoidal category and apply the construction in Definition (2.1) to several copies of this category.

Definition 3.1. *Let $\mathcal{F}$ be the category with objects n and morphisms given by*

$$\mathrm{Hom}_{\mathcal{C}}(n, m) = \begin{cases} \{1\} & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$$

This is a monoidal category with tensor product given by $n \otimes m = n + m$.

The main property of this example is that the object 1 in the monoidal category $\mathcal{F}$ is a universal object in a monoidal category. This means that if V is an object in a monoidal category $\mathcal{C}$ then there is a unique monoidal functor $\mathcal{F} \to \mathcal{C}$ which sends the object 1 to V. A consequence of this and the naturality of the product in Definition (2.1) is that if

we have objects V_1 and V_2 in monoidal categories $\mathcal{C}_1$ and $\mathcal{C}_2$ then we have a monoidal functor

$$\mathcal{F} \otimes \mathcal{F} \to \mathcal{C}_1 \otimes \mathcal{C}_2$$

Let $\mathbb{Z}\mathcal{F}$ be the free $\mathbb{Z}$-linear category on $\mathcal{F}$. Let ρ be a one dimensional representation of $\mathbb{Z}$ such that $\otimes^n \rho \neq 1$ for all $n > 0$. Then $\mathbb{Z}\mathcal{F}$ is also the category of invariant tensors associated to the representation ρ of $\mathbb{Z}$.

Then we consider the repeated product $\otimes^c \mathcal{F}$. This category is a groupoid. Let C be a set with c elements. The objects are sequences of elements of C. Given a sequence X, let $p\colon C \to \mathbb{N}$ be the function such that $p(c)$ is the number of times c appears in the sequence X. Then two sequences are isomorphic if and only if they have the same function. Also if two objects are isomorphic then there is only one isomorphism. One way to see this is to take the morphisms to be permutation diagrams with edges labelled by C and such that if two edges cross then they are labelled by distinct colours.

Now consider the repeated product $\otimes^c(\mathbb{Z}\mathcal{F})$ and take the free additive category on this repeated tensor product. Recall that the objects of the repeated product $\otimes^c(\mathbb{Z}\mathcal{F})$ are sequences of integers each labelled by a colour.

Definition 3.2. *Let V be the sum of the c sequences of length one consisting of the single integer 1. Consider the category of invariant tensors of V and let $\{F^{(c)}(n)\}$ be the associated tower of algebras.*

Given a function $p\colon C \to \mathbb{N}$ let $|p| = \sum_{c\in C} p(c)$.

Lemma 3.3. *For all $c \geq 1$ and $n \geq 0$ the algebra $F^{(c)}(n)$ is a direct sum of matrix algebras. The simple $F^{(c)}(n)$-modules are indexed by functions, p, such that $|p| = n$. The dimension of the simple module associated to p is the multinomial coefficient $\binom{|p|}{\{p(c)|c\in C\}}$.*

4. Bratteli diagrams

Let V be an object in a monoidal category $\mathcal{C}$, then associated to V is the category of invariant tensors and the tower of algebras. In this section we let V_1 be an object in $\mathcal{C}_1$ and V_2 an object in $\mathcal{C}_2$.

Definition 4.1. *Consider both V_1 and V_2 as objects in $\mathcal{C}_1 \otimes \mathcal{C}_2$ by taking them to be sequences of length one. Assume $\mathcal{C}_1 \otimes \mathcal{C}_2$ is additive and put $V = V_1 \oplus V_2$.*

Then we consider the category of invariant tensors associated to V and the tower of algebras associated to V.

First we compare $\mathcal{I}(V)$ with $\mathcal{I}(V_1) \otimes \mathcal{I}(V_2)$. The objects of the tensor product $\mathcal{I}(V_1) \otimes \mathcal{I}(V_2)$ are sequences where each term is of the form $\otimes^n V_1$ or $\otimes^n V_2$ for some $n \geq 0$. Let $\mathcal{I}$ be the full subcategory with objects those sequences in which every term is of the form V_1 or V_2.

Then by construction $\mathcal{I}$ is a full monoidal subcategory of $\mathcal{I}(V_1)\otimes\mathcal{I}(V_2)$. The connection between $\mathcal{I}$ and $\mathcal{I}(V)$ comes from noting that $\otimes^n V$ is the direct sum of of all objects of $\mathcal{I}$ which are sequences of length n.

Next we note that the inclusion functor $\mathcal{I} \to \mathcal{I}(V_1) \otimes \mathcal{I}(V_2)$ is an equivalence of categories. The inverse functor is constructed by writing a term $\otimes^n V_1$ by n terms V_1 and similarly for $\otimes^n V_2$.

The object V can be considered as an object of $\mathcal{C}_1\otimes\mathcal{C}_2$, $\mathcal{I}(V_1)\otimes\mathcal{I}(V_2)$ or $\mathcal{I}$. All these three cases give the same category of invariant tensors and the same tower of algebras. Let $A(n)$ be the endomorphism algebra of $\otimes^n V$ cosidered as an object in any of these categories. Another possibility is to consider V as an object of $\mathcal{A}(V_1) \otimes \mathcal{A}(V_2)$.

Definition 4.2. *Define $\widehat{A}(n)$ to be the endomorphism algebra of $\otimes^n V$ considered as an object of $\mathcal{A}(V_1) \otimes \mathcal{A}(V_2)$.*

Then, by construction, we have inclusions $\widehat{A}(n) \to A(n)$ for all $n \geq 0$. If the inclusions $\mathcal{A}(V_1) \to \mathcal{I}(V_1)$ and $\mathcal{A}(V_2) \to \mathcal{I}(V_2)$ are both isomorphisms then these inclusions will be isomorphisms. However, in general, $\widehat{A}(n)$ will be a proper subalgebra of $A(n)$.

Here we consider this tower of subalgebras. The main result is:

Proposition 4.3. *For $n \geq 0$, the algebra $\widehat{A}(n)$ is isomorphic to*

$$\bigoplus_{r+s=n} M(\binom{n}{r,s}) \otimes A_1(r) \otimes A_2(s)$$

where $M(N)$ is the algebra of $N \times N$ matrices.

Proof. First note that the inclusion $F^{(2)}(n) \to \widehat{A}(n)$ is an inclusion

$$\bigoplus_{r+s=n} M(\binom{n}{r,s}) \to \widehat{A}(n)$$

Now let e be a diagonal elementary matrix in $M(\binom{n}{r,s})$, so e is an idempotent permutation diagram. Then observe that we have an isomorphism

$$e\widehat{A}(n)e \cong A_1(r) \otimes A_2(s)$$

These idempotents give a decomposition of the identity on both sides of (4.3) so the result follows since both of these decompositions give the same Pierce decomposition. □

In this section we consider the Bratteli diagrams of the towers of algebras we are considering. The Bratteli diagram was introduced in [Bra72] in the study of approximately finite C^*-algebras. Assume we are given a tower of algebras such that each algebra is a direct sum of matrix algebras. Then associated to this tower of algebras is a graded directed graph called the Bratteli diagram. This has the following properties. The vertices of degree n correspond to the (isomorphism classes of) simple $A_1(n)$-modules; so, there is a single vertex v_0 of degree

0. Let v be a vertex of degree n associated to the simple module M. Then the dimension of M is the number of directed paths from v_0 to v.

Definition 4.4. *Next let Γ_1 be a directed graph with edge set E_1, vertex set V_1 and maps $h_1, t_1 \colon E_1 \to V_1$. Let Γ_2 be second directed graph. Then we define the product to have edge set $E = (V_1 \times E_2) \cup (E_1 \times V_2)$, vertex set $V = V_1 \times V_2$ and define $h, t \colon E \to V$ by*

$$\begin{aligned} h(e,v) &= (h_1(e), v) \quad & h(v,e) &= (v, h_2(v)) \\ t(e,v) &= (t_1(e), v) \quad & t(v,e) &= (v, t_2(v)) \end{aligned}$$

Note that this product is also associative and symmetric. Furthermore if Γ_1 and Γ_2 have basepoints v_1 and v_2 then we take (v_1, v_2) to be the basepoint in Γ.

Assume that the two towers $\{A_1(n)\}$ and $\{A_2(n)\}$ both have Bratteli diagrams. Then the tower of algebras $\{\widehat{A}(n)\}$ also has a Bratteli diagram and this Bratteli diagram is the product of the two Bratteli diagrams given in Definition (4.4).

This observation and Proposition (4.3) both give the following dimension formula. An irreducible representation, W, of $\widehat{A}(n)$ is labelled by a pair (W_1, W_2) where W_1 is an irreducible representation of $A_1(r)$ and W_2 is an irreducible representation of $A_2(s)$ where $r+s=n$. Then the dimension of W is given by

$$\dim(W) = \binom{n}{r, s} \dim(W_1) \dim(W_2) \tag{6}$$

There are two examples in which the inclusions $\widehat{A}(n) \to A(n)$ are isomorphisms. The first is the basic example of the object 1 in $\mathbb{Z}\mathcal{F}$. Then this construction shows that the Bratteli diagram for the tower of algebras $\{F^{(2)}(n)\}$ is Pascal's triangle. More generally, the Bratteli diagram for the tower of algebras $\{F^{(c)}(n)\}$ is the generalisation of Pascal's triangle which gives the multinomial coefficients. This directed graph has vertices $\mathbb{N}^c$ with directed edges given by increasing a single coordinate by 1 and the basepoint is the origin.

For the second example we interpret the Hecke algebras as endomorphism algebras of tensor powers of some object in a monoidal category. This can be achieved as follows. Let R be the ring defined by

$$R = \mathbb{Z}[\delta, q, z, 1/qz] / \langle \delta(q - q^{-1}) = (z - z^{-1}) \rangle$$

Then let $\mathcal{H}$ be the R-linear monoidal category obtained by taking the free R-linear category on the category of oriented tangles and imposing the HOMFLY skein relations, see [Jon87]. Then let V be the object whose identity morphism is a single descending string. Then the Hecke algebra $H(n)$ is the endomorphism algebra of $\otimes^n V$. Then in this example we also have that the inclusion $\widehat{H}(n) \to H(n)$ is an isomorphism. Then this construction shows that the Bratteli diagram for the tower of

algebras $\{H^{(c)}(n)\}$ is the Bratteli diagram for the Ariki-Koike algebras given in [AK94].

5. Symmetry

Let V be an object of $\mathcal{C}$. Then V can be regarded as an object in $\otimes^c\mathcal{C}$ in c different ways and we let $V^{(c)}$ be the direct sum of these. For $c = 2$ this object is given by taking $V_1 = V_2$ in Definition (4.1). Let $\{A^{(c)}(n)\}$ be the tower of algebras associated to $V^{(c)}$. For example, if we take V to be the object 1 of $\mathbb{Z}\mathcal{F}$ then this construction gives the algebras in Definition (3.2). Then the tower of algebras has an action of the symmetric group S_c which arises from the symmetric structure on the product in Definition (2.1). Let $A^{S(c)}(n)$ be the tower of algebras obtained by taking the fixed point subsets. Then the symmetric group acts freely so we have

$$\dim A^{S(c)}(n) = \frac{1}{c!}\dim A^{(c)}(n) \tag{7}$$

for all $n > 0$.

Assume V is a representation of a Hopf algebra H. Then $V^{(c)}$ is the representation of $\otimes^c H$ given by

$$\bigoplus_{i=1}^{c} \overbrace{1\otimes\ldots\otimes 1}^{i-1 \text{ factors}} \otimes V \otimes \overbrace{1\otimes\ldots\otimes 1}^{c-i \text{ factors}}$$

Then, for $n \geq 0$, the algebra $A^{(c)}(n)$ is the endomorphism algebra of $\otimes^n V^{(c)}$. The tower of algebras $\{A^{S(c)}(n)\}$ also has an interpretation as the tower associated to a representation of a Hopf algebra. This interpretation is given by using the same representation $V^{(c)}$ and regarding it as a representation of the wreath product $S_c \wr H$.

Next we give a generalisation of the construction (5). Let $\mathcal{C}$ be a K-linear monoidal category and assume $\mathcal{C}\otimes\mathcal{C}$ is additive. Then there is a functor $\mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ which on objects is given by $V \mapsto V^{(2)}$ and which on morphisms is given by $\phi \mapsto \phi\oplus\phi$.

In particular we have algebra homomorphisms

$$A(n) \to A^{(c)}(n)$$

These factor through $A^{S(c)}(n)$ to give algebra homomorphisms

$$A(n) \to A^{S(c)}(n)$$

Next we discuss the representation theory of the algebras $\{A^{S(2)}(n)\}$. Let s be the involution of $A^{(2)}(n)$ whose fixed point set is $\{A^{S(2)}(n)\}$. Then s induces an involution on the representations of $\{A^{(2)}(n)\}$. Let W be an irreducible representation of $\{A^{S(2)}(n)\}$. Then if W and $s(W)$ are not isomorphic W can be regarded as an irreducible representation of $\{A^{S(2)}(n)\}$; and if W and $s(W)$ are isomorphic W is the direct sum of two irreducible representations of $\{A^{S(2)}(n)\}$ of equal dimension.

$$
\begin{array}{ccccccccc}
1 & & & & & & & & \\
 & 1 & & & & & & & \\
1,1 & & 1 & & & & & & \\
 & 3 & & 1 & & & & & \\
3,3 & & 4 & & 1 & & & & \\
 & 10 & & 5 & & 1 & & & \\
10,10 & & 15 & & 6 & & 1 & & \\
 & 35 & & 21 & & 7 & & 1 & \\
35,35 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

FIGURE 1. Bratteli diagram of $\{F^{S(2)}(n)\}$.

Now assume that $\{A^{(2)}(n)\}$ is a direct sum of matrix algebras so that the dimension of $\{A^{(2)}(n)\}$ is the sum of the squares of the dimensions of the irreducible representations. Then $\{A^{(2)}(n)\}$ is also a direct sum of matrix algebras and the sum of the squares of the dimensions is half the dimension of $\{A^{(2)}(n)\}$ which is consistent with (7).

As an example of this we consider the tower of algebras $F^{S(2)}(n)$. The Bratteli diagram for these algebras is given in Figure 1.

This is essentially Pascal's triangle folded about the central axis. This is also the Bratteli diagram for the Temperley-Lieb algebras of type D_n (see [tD98] and [Fan97]). Each two part partition of n can be written as $(n-p,p)$ where $0 \leq 2p \leq n$. Corresponding to each such partition there is a representation of the n-string algebra whose dimension is given by the binomial coefficient $\binom{n}{p}$. These representations are irreducible except if $2p = n$ in which case they are the sum of irreducible representations of the same dimension.

Definition 5.1. *Let H be the group algebra of the wreath product $\mathbb{Z} \wr S_2$. This is a Hopf algebra. The algebra is obtained from the ring of Laurent polynomials $K[q_1, q_2, 1/q_1q_2]$ by adjoining an element σ which satisfies the relations*

$$\sigma^2 = 1 \qquad \sigma q_1 = q_2 \sigma \qquad \sigma q_2 = q_1 \sigma$$

The coproduct is given by

$$\Delta(q_1) = q_1 \otimes q_1 \qquad \Delta(q_2) = q_2 \otimes q_2 \qquad \Delta(\sigma) = \sigma \otimes \sigma$$

and the antipode is given by

$$S(q_1) = q_1^{-1} \qquad S(q_2) = q_2^{-1} \qquad S(\sigma) = \sigma$$

Note that we can eliminate the generator q_2 using $q_2 = \sigma q_1 \sigma$. Then the defining relations become $\sigma^2 = 1$ and $\sigma q_1 \sigma q_1 = q_1 \sigma q_1 \sigma$.

Definition 5.2. *Let x be an invertible scalar. Then we define a two dimensional representation $\rho^{(n)}$ for $n \geq 0$ by*

$$q_1 \mapsto \begin{pmatrix} x^n & 0 \\ 0 & 1 \end{pmatrix} q_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & x^n \end{pmatrix} \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

Then $\rho^{(n)}$ is irreducible if $x^n \neq 1$ and $\rho^{(0)}$ is the sum of two one dimensional representations. Denote these by ρ_+ and ρ_-. The tensor product of these representations with $\rho^{(1)}$ are given by the following two Lemmas.

Lemma 5.3. *For all $n > 0$,*

$$\rho^{(1)} \otimes \rho^{(n)} = \rho^{(n-1)} \oplus \rho^{(n+1)}$$

Proof. Let the ordered basis of $\rho^{(n)}$ used in (8) be $(e_1^{(n)}, e_2^{(n)})$. Consider the tensor product representation. The action on the subspace with ordered basis $(e_1^{(1)} \otimes e_1^{(n)}, e_2^{(1)} \otimes e_2^{(n)})$ is the representation $\rho^{(n+1)}$ and the action on the subspace with ordered basis $(\frac{1}{x} e_2^{(1)} \otimes e_1^{(n)}, \frac{1}{x} e_1^{(1)} \otimes e_2^{(n)})$ is the representation $\rho^{(n-1)}$. □

Lemma 5.4.

$$\rho^{(1)} \otimes \rho_{\pm} = \rho^{(1)}$$

Proof. The representation ρ_+ is the trivial representation so this case is clear. The representation $\rho^{(1)} \otimes \rho_-$ is given by taking the matrices in (8) for $n = 1$ and multiplying the matrix representing σ by -1. Let the ordered basis of this representation be (f_1, f_2). Then if we change to the ordered basis $(f_1, -f_2)$ we get the representation $\rho^{(1)}$. □

These tensor product decompositions show that if $x^n \neq 1$ for $n > 0$ then the tower of algebras given by $\mathrm{End}_H(\otimes^n \rho^{(1)})$ has the Bratteli diagram given in Figure 1.

6. Dilute Temperley-Lieb

In this section we consider the c-colour Temperley-Lieb algebras, $T^{(c)}(n)$. Then it is clear from the diagram point of view that each $T^{(c)}(n)$ is a cellular algebra in the sense of [GL96]. Next we discuss this cell structure in more detail, following [GM03]. Each diagram $D \in T^{(c)}(n)$ has say $|p|$ propagating strings. Then these $|p|$ strings are coloured, so, say $p(c)$ are coloured by $c \in C$. Then p is a function $p \colon C \to \mathbb{N}$. Define a partial order on these functions by $p_1 \leq p_2$ if $p_1(c) \leq p_2(c)$ for all $c \in C$. Then let $I(p)$ be the subspace with basis all diagrams, D, such that $p(D) \leq p$. Then for each p, $I(p)$ is an ideal and $I(p_1) \subset I(p_2)$ if and only if $p_1 \leq p_2$.

The algebra $F^{(c)}(n)$ is also a quotient of $T^{(c)}(n)$. The tower of algebras, $\{T^{(c)}(n)\}$, is obtained from the tower $\{F^{(c)}(n)\}$ by a Jones tower construction. This construction is described in [GdlHJ89, Chapter 2] and [HR, §4]. The Bratteli diagram for the tower of algebras $F^{(c)}(n)$

is a directed graph with vertices $\mathbb{N}^c$. Then we obtain the Bratteli diagram for the tower of algebras $T^{(c)}(n)$ by taking paths in the underlying undirected graph. In particular, the irreducible representations of $T^{(c)}(n)$ are indexed by sequences $(k_1,\dots,k_c)$ such that $k_i \geq 0$ and $n-k_1-\cdots k_c$ is even and non-negative. For the two-colour case, these paths are counted in [GKS92] and [Guy00].

The dimensions of the two colour algebras are given by

0	1	2	3
1	2	10	70

The simplest description of these numbers is the formula

$$\dim T^{(2)}(n) = C(n)C(n+1) \tag{9}$$

where $C(n)$ is the Catalan number. For c colours we have the formula

$$\dim T^{(c)}(n) = \sum_{n_1+\cdots n_c=n} \binom{2n}{2n_1,\dots,2n_c} \prod_{i=1}^{c} C(n_i)$$

Equivalently, in terms of exponential generating functions,

$$\sum_{n\geq 0} \frac{\dim T^{(c)}(n)}{(2n)!} z^{2n} = \left(\sum_{n\geq 0} \frac{C(n)}{(2n)!} z^{2n} \right)^c$$

This can be extended to the dimensions of the irreducible representations. let $F(x,y)$ be defined by

$$F(x,y) = \sum_{n,p\geq 0} \frac{(n-2p+1)}{p!(n-p+1)!} x^n y^{n-2p}$$

Let $S(n;k_1,\dots,k_c)$ be the simple $T^{(c)}(n)$-module associated to the vector $(k_1,\dots,k_c)$ and define

$$G(x;y_1,\dots,y_c) = \sum_{n,k_1,\dots,k_c} \dim S(n,k_1,\dots,k_c) \frac{x^n}{n!} y_1^{k_1} \dots y_c^{k_c}$$

Then these are related by

$$G(x;y_1,\dots,y_c) = \prod_{i=1}^{c} F(x,y_i)$$

In this rest of this section we restrict attention to just two colours and we also assume that the two parameters $\{\delta(c)|c \in C\}$ are both equal and we denote them both by δ. Then we let $T^{S(2)}(n)$ be the fixed point subalgebra of the involution which interchanges the two colours.

The general discussion in §5 applies and gives the following description of the irreducible representations of $T^{S(2)}(n)$. Let $S(n;r,s)$ be an irreducible representation of $T^{(2)}(n)$. Then if $r \neq s$ the restriction to $T^{S(2)}(n)$ is irreducible and the restrictions of $S(n;r,s)$ and $S(n;s,r)$ are isomorphic. The restriction of each representation $S(n,r,r)$ is the direct sum of two irreducible representations of the same dimension.

Definition 6.1. *Define the following elements of $F^{(2)}(n)$. The element $\pi_i(c)$ is the sum over all colourings of the identity permutation with the string i coloured by c. The element $s_i(c,d)$ is the sum over all colourings of the permutation $(i, i{+}1)$ such that the string from i to $i{+}1$ is coloured c and the string from $i+1$ to i is coloured d.*

Definition 6.2. *Define the following elements of $T^{(2)}(n)$. Let U_i be the diagram for the standard generators of the Temperley-Lieb algebras. Then the element $u_i(c,d)$ is obtained by summing over all colourings such that the top arc is coloured c and the lower arc is coloured d.*

Definition 6.3. *For $n > 1$, define the following elements of $T^{S(2)}(n)$,*

$$\begin{aligned}
e_i &= \pi_i(1)\pi_i(1) + \pi_i(2)\pi_i(2) \\
f_i &= \pi_i(1)\pi_i(2) + \pi_i(2)\pi_i(1) \\
s_i &= \sigma_i(1,2) + \sigma_i(2,1) \\
u_i &= u_i(1,1) + u_i(2,2) \\
t_i &= u_i(1,2) + u_i(2,1)
\end{aligned}$$

Then these elements generate $T^{S(2)}(n)$. For fixed i, the five dimensional algebra with these elements as basis also has a basis given by the following five orthogonal idempotents:

$$e_i - \frac{1}{\delta}u_i \qquad \frac{1}{2}(f_i \pm s_i) \qquad \frac{1}{2\delta}(u_i \pm t_i) \tag{10}$$

In order to describe the R-matrices and the braid matrices we extend the ring of scalars from the polynomial ring $\mathbb{Z}[\delta]$ to the ring of Laurent polynomials $\mathbb{Z}[q, 1/q]$ using the ring homomorphism determined by

$$\delta \mapsto -q^2 - q^{-2} \tag{11}$$

The construction (5) shows that the algebras $\{T^{S(2)}(n)\}$ can be used to construct an invariant of unoriented framed links just as the Temperley-Lieb algebras can be used to construct the Kauffman bracket polynomial. This invariant has the following description. First, for a link L let $\langle L\rangle$ be the Kauffman bracket normalised so that the empty link has the invariant 1. Then this invariant is multiplicative under disjoint union. Then the two-colour dilute version of this invariant is

$$\sum_{L=L_1\cup L_2} \langle L_1\rangle\langle L_2\rangle$$

The sum is over all ways of colouring the components by the two colours 1 and 2; and for a given colouring L_1 is the sublink coloured 1 and L_2 is the sublink coloured 2. Although they are different invariants, there are some similiarities with the link invariants in [Rol91]. Next we show that this invariant can be calculated by taking a Markov trace on the algebras $\{T^{S(2)}(n)\}$.

Proposition 6.4. *The tower of algebras* $\{T^{S(2)}(n)\}$ *has the property that*

$$T^{S(2)}(n+1) = T^{S(2)}(n) + T^{S(2)}(n)e_nT^{S(2)}(n) + T^{S(2)}(n)s_nT^{S(2)}(n) + T^{S(2)}(n)u_nT^{S(2)}(n) + T^{S(2)}(n)t_nT^{S(2)}(n) \tag{12}$$

Proof. First we check that the necessary condition in [Wes97] is satisfied. Then this can be shown by writing down a long, but finite, list of relations. □

The map of the braid group given in (5) is

$$\sigma_i^{\pm 1} \mapsto q^{\pm 2}e_i + u_i - Q^{\pm 1}s_i \tag{13}$$

where Q is an independent parameter. This parameter arises by taking into account the fact that the defining relations of the braid group are homogeneous so any representation can be scaled.

The map of the Temperley-Lieb algebra $T(n) \to T^{S(2)}(n)$ is given by

$$U_i \mapsto u_i + t_i$$

Note that the value of a closed loop in $T(n)$ is 2δ. There is a conditional expectation $\varepsilon_n \colon T^{S(2)}(n+1) \to T^{S(2)}(n)$ This conditional expectation is determined by

$$U_{n+1}aU_{n+1} = \varepsilon_n(a)U_{n+1}$$

It follows from (12) that this is determined by

$$\varepsilon_n(a) = \delta a, \varepsilon_n(ae_nb) = \delta ab, \varepsilon_n(as_nb) = 0, \varepsilon_n(au_nb) = ab, \varepsilon_n(at_nb) = 0$$

for all $a, b \in T(n)$. This give a sequences of traces $\tau_n \colon T(n) \to \mathbb{Z}[\delta]$ which are determined by $\tau_{n+1}(a) = \tau_n(\varepsilon_n(a))$ for all $a \in T(n+1)$. This sequence of traces gives a sequence of traces of the braid group algebras which satisfies the Markov property since we have

$$\varepsilon_n(a\sigma_n^{\pm 1}b) = (q^{\pm 2}\delta + 1)ab = -q^{\pm 4}ab$$

using (11).

The Yang-Baxter equation is the equation

$$R_i(u)R_{i+1}(uv)R_i(v) = R_{i+1}(v)R_i(uv)R_{i+1}(u) \tag{14}$$

for $|i-j| > 1$ and all u and v. The solution to the Yang-Baxter equation is given in [WN93] and [GP93, (4.34)].

Introduce the notation

$$[ax+b] = \frac{q^bu^a - q^{-b}u^{-a}}{q - q^{-1}}$$

This R-matrix is given in terms of the elements in Definition (6.3) by:

$$R_i(u) = [1-x][3-x]e_i + [3-x]f_i - [x][2-x]u_i + [x]t_i + [x][3-x]s_i \tag{15}$$

The relation (14) can be checked by checking it each irreducible representation of $T^{S(2)}(3)$. These have dimension 1, 3 and 5.

This solution has a number of properties. The first is that

$$R_i(1) = [3]$$

Another property is that taking the coefficients of $u^{\pm 2}$ gives a representation of the braid group. These coefficients are given by

$$\frac{q^{\mp 2}}{(q-q^{-1})^2}\sigma_i^{\mp 1}$$

where $\sigma_i^{\pm 1}$ is given by (13) with $Q = q$.

Another property of this R-matrix is that it has crossing symmetry. This means that it is invariant under the involution

$$x \leftrightarrow 3 - x \quad u_i \leftrightarrow e_i \quad t_i \leftrightarrow f_i \quad s_i \leftrightarrow s_i$$

The subalgebra generated by the braid group is the subalgebra generated by $\{e_i, u_i, s_i\}$. This subalgebra is not closed under the crossing symmetry and so the crossing symmetry implies that the R-matrix cannot be written as a polynomial in the braid matrix.

This R-matrix can also be written in terms of the basis of orthogonal idempotents in (10) as follows:

$$\begin{aligned}
- u^2q^2(q^2-1)^2R_i(u) = \\
(u-q^3)(u+q)(uq-1)(uq^3+1)\left[\frac{1}{2\delta}(u_i - t_i)\right] \\
(u-q^3)(u+q)(uq-1)(q^3+u)\left[\frac{1}{2}(f_i + s_i)\right] \\
(u-q^3)(u+q)(q-u)(q^3+u)\left[e_i - \frac{1}{\delta}u_i\right] \\
(u-q^3)(1+uq)(q-u)(q^3+u)\left[\frac{1}{2}(f_i - s_i)\right] \\
(1-uq^3)(1+uq)(q-u)(q^3+u)\left[\frac{1}{2\delta}(u_i + t_i)\right]
\end{aligned}$$

In particular, the idempotents are independent of the spectral parameter and so this R-matrix also has the property that

$$R_i(u)R_i(v) = R_i(v)R_i(u) \tag{16}$$

for all u and v.

References

[AK94] Susumu Ariki and Kazuhiko Koike. A Hecke algebra of $(\mathbf{Z}/r\mathbf{Z}) \wr S_n$ and construction of its irreducible representations. *Adv. Math.*, 106(2):216–243, 1994.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[Bra72] Ola Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

[CdM96] Arjeh M. Cohen and Ronald de Man. Computational evidence for Deligne's conjecture regarding exceptional Lie groups. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(5):427–432, 1996.

[DdM96] Pierre Deligne and Ronald de Man. La série exceptionnelle de groupes de Lie. II. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(6):577–582, 1996.

[Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.

[Del96] Pierre Deligne. La série exceptionnelle de groupes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(4):321–326, 1996.

[DPT03] Patrick Dorey, Andrew Pocklington, and Roberto Tateo. Integrable aspects of the scaling q-state Potts models. I. Bound states and bootstrap closure. *Nuclear Phys. B*, 661(3):425–463, 2003, arXiv:hep-th/0208111.

[Fan97] C. Kenneth Fan. Structure of a Hecke algebra quotient. *J. Amer. Math. Soc.*, 10(1):139–167, 1997.

[GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F. R. Jones. *Coxeter graphs and towers of algebras*, volume 14 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, New York, 1989.

[GKS92] Richard K. Guy, C. Krattenthaler, and Bruce E. Sagan. Lattice paths, reflections, & dimension-changing bijections. *Ars Combin.*, 34:3–15, 1992.

[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[GM03] Uwe Grimm and Paul P. Martin. The Bubble Algebra: Structure of a Two-Colour Temperley-Lieb Algebra. *J. Phys. A: Math. Gen.*, 36:10551–10571, 2003, arXiv:math-ph/0307017.

[GP93] Uwe Grimm and Paul A. Pearce. Multi-colour braid-monoid algebras. *J. Phys. A*, 26(24):7435–7459, 1993.

[Gri94] Uwe Grimm. Dilute Birman-Wenzl-Murakami algebra and $D^{(2)}_{n+1}$ models. *J. Phys. A*, 27(17):5897–5905, 1994.

[Guy00] Richard K. Guy. Catwalks, sandsteps and Pascal pyramids. *J. Integer Seq.*, 3(1):Article 00.1.6, 1 HTML document (electronic), 2000.

[HR] Tom Halverson and Arun Ram. Partition Algebras, arXiv:math.RT/0401314.

[Jon87] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)*, 126(2):335–388, 1987.

[Roc92] Ph. Roche. On the construction of integrable dilute *ade* models. *Phys. Lett. B*, 285(1-2):49–53, 1992.

[Rol91] Dale Rolfsen. PL link isotopy, essential knotting and quotients of polynomials. *Canad. Math. Bull.*, 34(4):536–541, 1991.

[tD98] Tammo tom Dieck. Temperley-Lieb algebras associated to the root system *D*. *Arch. Math. (Basel)*, 71(5):407–416, 1998.

[Wes95] B. W. Westbury. The representation theory of the Temperley-Lieb algebras. *Math. Z.*, 219(4):539–565, 1995.

[Wes97] Bruce W. Westbury. Quotients of the braid group algebras. *Topology Appl.*, 78(1-2):187–199, 1997.

[WN93] S. Ole Warnaar and Bernard Nienhuis. Solvable lattice models labelled by Dynkin diagrams. *J. Phys. A*, 26(10):2301–2316, 1993.

Mathematics Institute, University of Warwick, Coventry CV4 7AL
E-mail address: `bww@maths.warwick.ac.uk`